THE PROBLEM OF TAKING INTO ACCOUNT THE SCALE FACTOR IN THE MECHANICS OF A SOLID DEFORMED BODY

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It is known that the macroscopic properties of a solid body depend on its absolute dimensions. The basis for such an assertion is the nature of the interaction of particles (i.e., atoms, molecules or molecular groups) in a solid body [1]. The particles at the surface experience a one-sided action from the side of other particles in the body, while in the bulk of the body there is a statistical symmetry of the forces with which the particles interact. From the macroscopic point of view of the mechanical properties of an isotropic solid body, this must lead to considerable nonuniformity near the boundary and to surface tension.

The coefficient of surface tension for solid bodies is of the order of 10^{-3} kgf/cm [2], i.e., in problems which can be solved within the framework of physical and geometrical linearity, the effect of surface tension can be neglected due to its smallness. In what follows, we will study only surface nonuniformity, assuming that far from the boundary, the solid is uniform.

Under these assumptions, an experimental investigation [3] of thin polymer filmer, specimens of which in dynamic tests are viewed as three-layer plates, is undertaken. It is shown that the dynamic modulus of boundary layers of the film exceeds the modulus of the central layer by a factor of 2-3. The thickness of the boundary layer is on the order of 10^{-3} mm. The error in the measurements is estimated at 30%.

The use of films as specimens in such studies raises some doubt as to the validity of the identification of the phenomenon examined from the results of the experiments [3]. Indeed, the thickness of the film depends on the thermal force parameters of the technological process with which the film was prepared. The mechanical properties of the material, due to its orientational stretching, depends strongly on these parameters. The fact that in this case it is not so much the scale factor as the technological factor that is observed has not been excluded.

1. In order to clarify the effects of surface nonuniformity in a solid body, we will examine the behavior of specimens that differ only in geometry. The thermal force histories of these specimens, including also the technological preparation, are practically identical. In this case, all dimensions of the specimen are large in comparison with the expected thickness of the surface layer.

The surface nonuniformity is estimated on prismatic specimens of polymethyl methacrylate (PMMC) and AMTs aluminum alloy.

The length of the specimens is 320 mm, while the dimensions of the transverse cross section are not less than 10 mm. A system of clamps [4] ensures central stretching of the specimens, while a differential measuring scheme [5] permits using observations of the fourth significant figure in the signal from the sensor in the numerical analysis of the results. The strain sensors consist of FKPA 5-50 Kh strain resistors, which are accumulated on the lateral surface of the specimen. The average integral, along the base of the sensor, components of the stresses ε_{xx}, ε_{yy}, ε_{zz}, where z is the longitudinal axis of the specimen, x and y are the symmetry axes of the transverse section of the specimen, are measured.

The differential scheme for the measurements [5] determines the difference $(\varepsilon_{xx} - \varepsilon_{yy})$. As tests show, for specimens with a square transverse section (10×10 mm^2), the complex

$$\omega = 2\left|\frac{\varepsilon_{xx} - \varepsilon_{yy}}{\varepsilon_{xx} + \varepsilon_{yy}}\right|$$

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does not exceed, with the exception of a single specimen, the magnitude 0.001. For the exceptional specimen, $\omega = 0.01$. A test of specimens with a rectangular section (25×10 mm^2) gave

$$\omega = 0.026 \pm 0.002 \text{ for PMMA}, \quad \omega = 0 \pm 0.001 \text{ for AMTs}.$$

Here, $|\varepsilon_{xx}| \leqslant 0.4 \cdot 10^{-3}$ and $|\varepsilon_{xx}| \geqslant |\varepsilon_{yy}|$, i.e., the transverse deformation of the wide boundary of the specimen exceeds the deformation of the narrow one. It should be noted that the surface tension qualitatively gives the same effect: the rectangular transverse section of the specimen on deformation transforms into a square one.

The phenomenon of surface nonuniformity is analyzed numerically assuming that the modulus decreases exponentially with distance from the surface. For the case that the modulus on the surface is two times greater than the modulus in the bulk of the specimen, the dependence of ω on the layer thickness has the following form:

ω	0.056	0.022	0.010	0.002
s, mm	0.50	0.25	0.12	0.06

s is the distance from the surface, where the modulus is 40% less than the maximum value. Thus, in our experiments, the thickness of the layer for PMMC is a quantity on the order of 0.1 mm; for the AMTs alloy, the effect of surface nonuniformity was not observed due to its smallness.

This result can be predicted if it is assumed that the thickness of the surface depends on the magnitude of the characteristic size of the elements in the structure of the solid body. For metals, this quantity is 1-5 orders of magnitude smaller than for polymers. Moreover, the elastic constants of the metal are practically independent of the past thermal force perturbations, which cannot be said of a polymer.

The experiment examined here shows a weak dependence of the results of the measurements on the effect being studied. In order to amplify this dependence, mechanical states with a large gradient in stress near the surface are necessary. Such states give a contact interaction. In order to verify this idea, we will examine the following problem in the theory of elasticity.

2. Let a stamp be axisymmetrically pressed into a three-layer elastic foundation with no friction in the region of contact $r \leqslant a$. The foundation consists of a rigid half space, on which lies a packet of two elastic layers with overall thickness $H^* = H + h$ without friction. The lower layer has thickness H and elastic characteristic G and ν, while the upper layer has a thickness h and elastic characteristics G_1 and ν. There is total coupling between the layers. It is assumed that $h \ll H$, $h \ll a$, and $G < G_1$.

Investigation of the problem indicated permits estimating the degree to which the surface nonuniformity affects the contact rigidity of the material, while varying the parameter $\lambda = H/a$ over a wide range allows a study of the scale factor.

It is shown in [6, 7] that with the assumptions made above the upper layer can be viewed as an elastic membrane with a tensile rigidity $k = 2hG_1(1-\nu)^{-1}$ and zero bending rigidity. The equations that characterize the deformation of a membrane in its plane under the action of tangential forces, applied to its boundary, have the following form for the axisymmetrical case:

$$k\left(\Delta u^* - \frac{u^*}{r^2}\right) = -\tau(r) \quad \left(\Delta = \frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr}\right), \tag{2.1}$$

where $u^*(r)$ is the displacement of points on the membrane in a radial direction. Taking into account (2.1) and the comments made above, the contact problem stated can be formulated as the following boundary-value problem for the Lamé equation:

$$\tau_{rz}(r, 0) = 0, \; W(r, 0) = 0 \quad \text{for} \quad r \geqslant a; \tag{2.2}$$

$$\sigma_z(r, H) = 0, \; W(r, H) = -[\delta - f(r)] \quad \text{for} \quad r < a; \tag{2.3}$$

$$k\left(\Delta u - \frac{u}{r^2}\right) = \tau_{rz} \quad \text{for} \quad z = H, \; 0 \leqslant r < \infty. \tag{2.4}$$

The stresses and displacements disappear for $r \to \infty$. Here, δ is the translational displacement of the stamp along the axis of symmetry under the action of the forces P applied to it; f(r) is a function describing the shape of the base of the stamp. In (2.4), we took into account the fact that $U^*(r) = u(r, H)$ and $\tau(t) = -\tau_{rz}(r, H)$. Using Hankel's integral transformation with respect to the variable r, the boundary-value problem (2.2)-(2.4) is reduced to an integral equation of the first kind relative to the function describing the distribution of contact pressures q(r), acting between the stamp and the base:

$$\int_0^a q(\rho) K\left(\frac{\rho}{H}, \frac{r}{H}\right) \rho d\rho = l\Theta H(\delta - f(r)), \quad r \leqslant a; \tag{2.5}$$

$$K(\tau, t) = \int_0^\infty L(u) J_0(u\tau) J_0(ut)\, du,$$
$$L(u) = \frac{u \sinh 2u - \frac{2}{\varkappa} u^2 + \frac{2l}{m} \sinh^2 u}{2u \cosh^2 u + \frac{1}{m}(\sinh 2u + 2u)}; \tag{2.6}$$

$$m = \frac{k}{\Theta H}, \ \Theta = \frac{G}{1-\nu}, \ l = \frac{(\varkappa+1)^2}{4\varkappa}, \ \varkappa = 3 - 4\nu. \tag{2.7}$$

Here $J_0(x)$ is Bessel's function, while the function L(u) has the following properties:

$$L(u) = Au + O(u^3) \quad \text{for} \quad u \to 0, \ A = \frac{\varkappa(m+l) - m}{(m+2)\varkappa}; \tag{2.8}$$

$$L(u) = 1 + \frac{B}{u} + O\left(\frac{1}{u^2}\right) \quad \text{for} \quad u \to \infty, \ B = \frac{l-1}{m}. \tag{2.9}$$

Let us transform in (2.5) to dimensionless variables using the formulas

$$\rho' = \rho/a, \ r' = r/a, \ \delta' = \delta/a, \ \varphi(\rho') = q(\rho), \ g(r') = (1/a)f(r),$$

then the integral equation of the problem takes the form (the primes are omitted in what follows)

$$\int_0^1 \rho\varphi(\rho) K\left(\frac{\rho}{\lambda}, \frac{r}{\lambda}\right) d\rho = l\Theta\lambda[\delta - g(r)], \quad r \leqslant 1. \tag{2.10}$$

We used the integrals

$$\int_0^\infty J_0(u, \tau) J_0(ut)\, du = \frac{2}{\pi(\tau+t)} K(l), \ l = \frac{2\sqrt{\tau t}}{\tau + t},$$

$$\int_0^\infty \left[J_0(u\tau) J_0(ut) - e^{-u/2}\right] \frac{du}{u} = \frac{1}{2}\left[\operatorname{sgn}(\tau - t) \ln\frac{\tau}{t} + \ln \tau t\right].$$

Let us represent the kernel in (2.6) taking into account (2.9) in the form

$$K(\tau, t) = \frac{2}{\pi(\tau+t)} K(l) - \frac{B}{2}\left[\operatorname{sgn}(\tau - t) \ln\frac{\tau}{t} + \ln \tau t\right] - F(\tau, t); \tag{2.11}$$

$$F(\tau, t) = \int_0^\infty \left\{\left[1 + \frac{B}{u} - L(u)\right] J_0(u\tau) J_0(ut) - \frac{B}{u} e^{-u/2}\right\} du. \tag{2.12}$$

Here, the function $F(\tau, t)$, at least is bounded for all $0 \leqslant \tau \leqslant 1/\lambda$, $0 \leqslant t \leqslant 1/\lambda$.

Substituting (2.11) and (2.10) and letting λ go to infinity, which corresponds (for a fixed a) to the case of degeneration of the lower elastic layer in the half space ($H = \infty$), we arrive at the following integral equation:

$$\int_0^1 \frac{\varphi(\rho)\rho}{\rho + r} K\left(\frac{2\sqrt{\rho r}}{\rho + r}\right) d\rho = \frac{\pi}{2} l\Theta[\delta - g(r)], \quad r \leqslant 1. \tag{2.13}$$

We note that without being aware of the existence in the material of a thin surface layer with special physicomechanical properties, it could be assumed that the integral equation (2.13) describes the axisymmetrical contact problem for a uniform elastic half space, for which the contact rigidity is $\Theta^* = G^*/(1 - \nu) = \Theta l$. For $\nu = 0.3$, the quantity Θ^* exceeds Θ

by 9%. Thus, the surface nonuniformity results here in an insignificant increase in the contact rigidity. As will be shown in what follows, the situation is different for very small λ, when the relative thickness of the lower layer approaches 0, but as before $H \to h$.

In order to investigate the case in very small λ, it is necessary to find the asymptotic expression of the kernel (2.6) for large τ and t. This can be done if in (2.6) L(u) is replaced by its asymptotic representation (2.8) for small u, using then the integral [8]

$$\gamma(\tau, t) = \int_0^\infty u J_0(u\tau) J_0(ut)\, du,$$

where the equality is understood in the sense of the theory of generalized functions, while $\gamma(\tau, t)$ represents a zero-order harmonic of the two-dimensional Dirac delta function. Thus, we verify that the degenerate, for very small λ, solution of the problem is determined by the integral equation

$$\int_0^1 \varphi(\rho)\, \gamma\left(\frac{\rho}{\lambda}, \frac{r}{\lambda}\right) \rho\, d\rho = l\Theta\lambda A^{-1}[\delta - g(r)] \quad (r \ll 1), \tag{2.14}$$

following from (2.10) with the transformations indicated above.

We note that the axisymmetrical contact problem for a uniform elastic layer with relative thickness λ is also reduced, for very small λ, to an integral equation of the type (2.14), where it is only necessary to set $l\Theta A^{-1} = 2\Theta^*$. Here, for $\nu = 0.3$, the quantity Θ^* is greater by 9 (m = 1), 18 (m = 10), and 25% (m = ∞). Thus, the surface nonuniformity results here in a greater contribution towards increasing the contact rigidity, which is in fact the manifestation of the scale factor.

3. The technical realization of the contact problem requires precise specification of the boundary conditions, since the displacement of the stamps must be measured with a resolution of 10^{-5} mm. This specification concerns not only the conditions for the force interaction in the contact zone, but also the deformation of stamps as structural elements of the setup.

The setup shown by the diagram in Fig. 1 permits a technical realization of the contact problem being examined. In accordance with Fig. 1, specimens 4 of the material being investigated are compressed by steel spheres 3, when diameters are 25 mm. The deviation of the surface of the steel sphere from an ideal sphere does not exceed 0.001 mm, while the height of the microirregularities of the surface does not exceed 10^{-4} mm. The spheres are joined with massive cylinders 2, when inner endface edges are fitted into the surface of the sphere.

Air, which passes through constant throttles 5, is introduced into the inner cavity of the cylinders. Under the action of the air pressure, the spheres compress the specimens, leaving the edges of the cylinders in doing so.

The manometers 1 permit recording the compressive forces of the specimens and the destruction of symmetry of the entire scheme, which in essence represents a pneumatic bridge, measuring the difference of the mutual displacements of the spheres in both arms of the bridge. The resolution of this scheme in measuring the displacement is 10^{-4} mm, the difference in the displacements with resolution 10^{-5} mm, and in measuring the forces 10^{-3} kgf. In constructing the setup, the signals from deformation of its elements and thermal effects are automatically compensated.

Thus, the setup examined here compares the deformation characteristics of two specimens, the thickness of which can vary from 10^{-3} to 10 mm. In experiments with PMMC specimens having a thickness of 1.77 and 2.37 mm, the effect of surface nonuniformity showed up as follows. For a force of P = 2 kgf the complex

$$\left(\frac{\delta_1}{\delta_2}\right)_t \Big/ \left(\frac{\delta_1}{\delta_2}\right)_e = 1.05 \pm 0.005, \tag{3.1}$$

where the index 1 represents a thin specimen; 2 represents a thick specimen; t denotes the theoretical estimate without taking into account the surface nonuniformity; e represents the experimental data. If there were no surface nonuniformity, then this complex would equal unity.

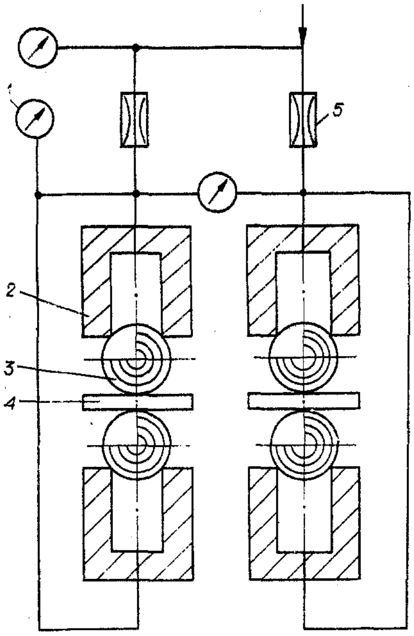

Fig. 1

It should be noted that the symmetry of the measurement scheme is established beforehand for the two identical specimens with a precision of ±0.005. Destruction of symmetry with different specimens is related not only to the difference in the specimen thicknesses, but also to the effect of the surface nonuniformity, which increases with decreasing specimen thickness. Complex (3.1) is formed so as to estimate quantitatively the effect of the surface nonuniformity with some uncertainty in the deformation characteristics of the specimens. Indeed, the effect of an error in the moduli on the final result (3.1) will be greatly decreased because it is not the displacements of the stamps themselves that are compared, but their ratio.

Let us calculate complex (3.1) theoretically, i.e., let us replace $(\delta_1/\delta_2)_e$ by the corresponding theoretical estimate taking into account the surface nonuniformity. For the case being examined, the radius of the compressing spheres is R = 1.25 cm, the compressing force is P = 2 kgf, the thickness of the specimens is $2H_1$ = 0.177 cm, $2H_2$ = 0.237 cm, and the elastic constants of the material of these spcimens are $E = 3 \cdot 10^4$ kg/cm^2 and $\nu = 0.32$. Without knowing the precise value of the constant m in formula (2.7), we will assume that it is equal to 10. Let us write out formulas [8] which give the limiting solution for small and large values of the parameter $\lambda = H/a$ for the problem of compression of a layer with thickness 2H by two, symmetrically placed, identical parabolic stamps. For small λ, we have

$$q(r) = (\Theta/HR)(a^2 - r^2),\ \delta = a^2/2R,\quad P = \pi\Theta a^4/2HR, \tag{3.2}$$

and for large λ

$$q(r) = (4\Theta/\pi R)\sqrt{a^2 - r^2},\quad \delta = a^2/R,\quad P = 8\Theta a^3/3R. \tag{3.3}$$

From Eqs. (3.2) and (3.3), we find, for small and large λ, respectively,

$$\lambda = \sqrt[4]{\frac{\pi\Theta H^3}{2RP}},\quad \delta = \sqrt{\frac{PH}{2\pi\Theta R}}; \tag{3.4}$$

$$\lambda = \sqrt[3]{\frac{8\Theta H^3}{3RP}},\quad \delta = \sqrt[3]{\frac{9P^2}{64\Theta^2 R}}. \tag{3.5}$$

Equation (3.4) ensures adequate accuracy if $\lambda \leqslant 2$, while Eq. (3.5) if $\lambda \geqslant 3$ [8]. For the case of a thin specimen, according to the first equation of (3.4), we obtain λ = 1.64 and for the case of a thick specimen, from the first equation, we obtain λ = 3.10. Thus, for calculating complex (3.1) in the case of thin and thick specimens, it is possible to use Eqs. (3.4) and (3.5), respectively. In this case, if the specimen is nonuniform in thickness, in (3.4) Θ should be replaced by $\Theta^* = l\Theta(2A)^{-1}$, while in (3.5) Θ should be replaced by $\Theta^* = \Theta l$. In calculating in this manner, the ratio δ_1/δ_2 for uniform and nonuniform specimens, with m = 10, we obtain the value 1.055 for complex (3.1). In the same manner, we verify that complex (3.1) is greater than 1 for all $1.43 \leqslant m < \infty$.

Thus, there exists a range of values of m, in which theory agrees qualitatively with experiment, and in this range there exists a value of m that ensures quantitative agreement.

We consider the results obtained here as a significant indication of the existence of surface nonuniformity in solid bodies.

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SELF-MODELING PROBLEMS IN THE DYNAMIC BENDING OF BEAMS

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The study of self-similar motions of continuous media is very fruitful [1]. The set of self-similar problems is limited by restrictions on the dimensionalities of the characteristic quantities. In those cases when these requirements are satisfied, the mathematical aspect of the problem can be greatly simplified.

In the present work, we examine the self-similar problems of dynamic bending of beams, satisfying the dynamic Bernoulli–Euler equation. For infinitely long beams, self-similar solutions and solutions including self-similar components are known [2-8]. All these solutions have been obtained, however, without using the properties of self-similarity. In the present work, we propose a general method that permits studying a wide class of self-similar solutions. Known self-similar solutions can be obtained as particular cases based on this method. In addition, methods are established for solving problems of bending of beams with moving supports, whose motion occurs within a regime that retains the self-similarity of the problem. We will refer to this regime as the self-similar regime of motion. The bending of a beam under the action of a force that moves along the beam in the self-similar regime indicated is investigated for the first time.

The properties of self-similarity were used previously in [9] for studying the deformation of membranes with movable boundaries.

1. Let us examine the equation for bending of a beam

$$EI\partial^4 w/\partial x^4 + m\partial^2 w/\partial t^2 = q\ (x, t), \quad (1.1)$$

where w is the deflection; t, time; x, coordinate; E, modulus of elasticity of the material; I, moment of inertia of a section; m, an adjustable mass; q, an adjustable load.

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